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A SOURCE SOLUTION FOR SHORT CRESTED WAVES

BY

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FOREWARD

This study was performed under two contracts, as it is basic to both. Consequently it is being issued as a report on both Series 3, and Series 61.

ABSTRACT

A Green's function for the boundary value problem arising in the diffraction of short-crested waves around obstacles of bounded cross section is presented. The diffraction problem is formulated in a precise way which assures the existence and uniqueness of a solution. The Green's function is so constructed as to make possible a representation of the velocity potential at internal points of the fluid in terms of its values on the obstacle, thus in general reducing the diffraction problem to the solution of a Friedholm integral equation of the second kind. Two problems of interest in the theory of surface waves, the production of waves by a moving partition, and the reflection from a horizontal strip are studied by means of the Green's function. Numerical results are obtained for the first problem and indications of numerical procedures given for the second. In particular, the strip problem is so formulated as to make possible the application of the variational methods of Schwinger.

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A SOURCE SOLUTION FOR SHORT CRESTED WAVES.

by

R. C. MacCamy

1. Introduction

A difficult but fundamental problem in the theory of surface water waves is the study of diffraction around fixed obstacles. Such a situation arises in the theory of docks, piles, breakwaters and, as an auxiliary problem, in the study of ship motion. In three dimensional motion the problem has been attacked by Fritz John[1] who succeeded in obtaining an integral equation formulation involving singular kernels. Beyond this little has been done for the case of obstacles of finite extent.

The purpose of the present report is to present a source solution in the case of the so-called short-crested waves, i.e., waves which are periodic in a direction perpendicular to that of propagation. Such a solution is the fundamental tool in an investigation of associated diffraction problems by means of Fritz John's methods. Physically, it gives the velocity potential of a line source on which the strength varies periodically along the line. Mathematically it enables one to express the potential of the interior of the fluid in terms of data on the obstacle surface.

In Section two we present the function and study its properties as well as indicate the general diffraction problems to which it is applicable. In the next two sections we indicate how the solution may be used in two problems, the determination of the waves produced by a wave-maker, and the reflection coefficient for a rigid horizontal barrier in the free surface. A few numerical results are given for the first problem.

The use of a source solution, or Green's function, is well known for other boundary value problems. In addition to yielding existence theorems which insure that the problem is correctly formulated, they may be used to obtain some approximate solutions. For example, it is shown in Section 4 that such a formulation enables one to invoke the variational procedures of Schwinger.

An effort has been made in this report to present ideas which admit of numerical computations. Some work of this kind is, in fact, underway now and will be presented at a later time.

2. The Green's Function

In order to introduce the source solution in a natural way we will discuss briefly the general diffraction problem for short-crested waves in the presence of cylindrical obstacles.

Suppose an incompressible, non-viscous fluid fills the region, $-\infty \leq x \leq +\infty$,

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* Superscript numerals refer to References at end of report.

$-\infty \leq Z \leq +\infty$, $0 < y < a$. The plane $y = 0$ is to be a rigid bottom and $y = a$ a free surface. We assume the motion to be generated initially by a short-crested wave system, of small amplitude, progressing in a positive x-direction. Such a system may be represented by its velocity potential,

$$\Phi^{(i)} = A \cosh \gamma_0 y\, e^{i(w_0 x - \sigma t)} \begin{matrix}\cos\\ \sin\end{matrix} kZ = \mathrm{Re}\left\{\phi^{(i)} e^{-i\sigma t}\right\} \begin{matrix}\cos\\ \sin\end{matrix} kZ \tag{2.1}$$

where A is a constant and γ_0, w_0 are determined by

$$K = \frac{\sigma^2}{g} = \gamma_0 \tanh \gamma_0 a, \qquad w_0^2 = \gamma_0^2 - k^2 \quad . \tag{2.2}$$

It should be noted that Equation (2.2) puts an upper limit on k in order that the waves progress in the x-direction, namely $k < K$, and we shall henceforth assume this to be the case.

Now let a cylindrical obstacle of bounded cross section be fixed in the fluid. The resulting motion once steady state is reached, will again be time periodic with frequency σ. Moreover, we reduce the problem to one in two-dimensions by assuming the Z dependence to remain in the form $\begin{matrix}\cos\\ \sin\end{matrix} kZ$. Under these assumptions we write for the velocity potential of the ensuing motion,

$$\Phi(x, y, Z, t) = \mathrm{Re}\left(\phi(x, y) e^{-i\sigma t}\right) \begin{matrix}\cos\\ \sin\end{matrix} kZ \tag{2.3}$$

and there results, for ϕ, the equation

$$\phi_{xx} + \phi_{yy} - k^2 \phi = 0 \quad \text{in fluid} \tag{2.4}$$

If we denote by C_0 the trace of the obstacle on the x-y plane, and by C_F that portion of $y = a$ exterior to C_0, we find that ϕ is also subject to the boundary conditions,

$$\phi_y - K\phi = 0 \quad \text{on} \quad C_F \tag{2.5}$$

$$\phi_y = 0 \quad \text{on} \quad y = 0 \tag{2.6}$$

$$\phi_n = 0 \quad \text{on} \quad C_0 \tag{2.7}$$

where n denotes the normal to C_0.

In order to fix the solution uniquely it is necessary to specify conditions at infinity. We would expect the motion to consist of an incident plus a reflected and transmitted wave, i.e., a condition of the form

$$\phi - \phi^{(i)} \to T e^{i w_0 x} \qquad x \to +\infty, \qquad \phi - \phi^{(i)} \to R e^{-i w_0 x} \quad \text{as} \quad x \to -\infty . \tag{2.8}$$

It turns out to be sufficient to demand only the weaker condition

$$\lim_{L \to \infty} \left\{ \int_0^a |\psi_x (L, y) - i C \psi (L,y)|^2 dy + \int_0^a |\psi_x (-L,y) - i C \psi (-L,y)|^2 dy \right\} = 0 \tag{2.9}$$

for some $C > 0$, where $\psi = \phi - \phi^{(i)}$ represents the scattered wave. It can be shown that there is a unique ϕ satisfying (2.4), (2.5), (2.6), (2.7) and (2.9) and that this solution does not indeed satisfy (2.8).

The Green's function, $G(x, y, x', y')$, which depends on a parameter point (x', y') is to be so constructed that it expresses values or solutions at (x', y') in terms of data on the obstacle surface C_0. It is to satisfy (2.5) on the entire $y = a$ plane, as well as (2.6), so as to suppress reference to values of the solution on $y = 0$ and C_F. In addition it must have a singularity of a certain type for $(x, y) = (x', y')$ in order that the operation

$$\int_{-\infty}^{+\infty} \int_0^a (G_{xx} + G_{yy} - k^2 G) \, \phi (x, y) \, dy \, dx$$

reproduce solutions of (2.4). Finally G must carry the behavior of the solution for large $|x|$ and hence must satisfy (2.9).

The fundamental singularity of Equation (2.4) is found by separation of variables and is given by $K_0 \left(k \sqrt{x^2 + y^2} \right)$ where K_0 is essentially the Hankel function of order 0 and first type with pure imaginary argument. For our purpose, the important properties of $K_0(Z)$ are

$$K_0 (Z) = O(e^{-z}) \qquad \text{for large} \quad Z , \tag{2.10}$$

$$K_0 (Z) = A(Z) \log \frac{1}{Z} + B(Z) \tag{2.11}$$

where $A(Z)$, $B(Z)$ are regular for real Z and $A(0) = 1$.

It can be shown that the Green's function as defined is unique and we proceed to give such a function. Consider,

$$G(x,y,x,y) = \frac{1}{\pi}\int_{0\,C}^{\infty} \frac{\cosh \gamma y' \{K \sinh \gamma(a-y) - \gamma \cosh \gamma(a-y)\}}{\gamma\{K \cosh \gamma a - \gamma \sinh \gamma a\}} \cos w\,|x-x'|\,dw \quad \text{for} \quad y \geq y'$$

$$= \frac{1}{\pi}\int_{0\,C}^{\infty} \frac{\cosh \gamma y \{K \sinh \gamma(a-y') - \gamma \cosh \gamma(a-y')\}}{\{K \cosh \gamma a - \gamma \sinh \gamma a\}} \cos w\,|x-x'|\,dw \quad \text{for} \quad y \leq y' \qquad (2.12)$$

where $\gamma^2 = w^2 + k^2$ and C is a contour consisting of the positive x-axis except for a small semi-circle around $w_0 = \sqrt{\gamma_0^2 - k^2}$ (which is real if $k < K$). This function was given by Heins(2) in its partial fraction expansion form.

Making use of the identities,

$$\frac{\cosh \gamma \xi \{K \sinh \gamma(a-\zeta) - \gamma \cosh \gamma(a-\zeta)\}}{\gamma\{K \cosh \gamma a - \gamma \sinh \gamma a\}} = \frac{-(K+\gamma)\cosh \gamma\xi \cosh \gamma\zeta\, e^{-\gamma a}}{\gamma\{K \cosh \gamma a - \gamma \sinh \gamma a\}} + \frac{e^{-\gamma(\zeta-\xi)}}{2\gamma} + \frac{e^{-\gamma(\zeta+\xi)}}{2\gamma},$$

$$\int_0^{\infty} \frac{e^{-\sqrt{w^2+k^2}\,b}}{\sqrt{w^2+k^2}} \cos wx\,dw = K_c\left(k\sqrt{x^2+b^2}\right), \qquad (2.13)$$

expression (2.12) may be transformed into,

$$G(x,y,x',y') = \frac{1}{2\pi} K_0\left(k\sqrt{(x-x')^2 \quad (y+y')^2}\right) + \frac{1}{2\pi} K_0\left(k\sqrt{(x-x')^2 \quad (y+y')^2}\right)$$

$$-\int_{0\,C}^{\infty} \frac{(K+\gamma)\cosh \gamma y \cosh \gamma y'\, e^{-\gamma a}}{\gamma\{K \cosh \gamma a - \gamma \sinh \gamma a\}} \cos w\,|x-x'|\,dw\,. \qquad (2.14)$$

We proceed to verify that. G has the desired properties. From (2.14) $G_y = 0$ on $y = 0$ and from (2.13) $G_y - KG = 0$ on $y = a$. Moreover, G satisfies (2.4) except at (x', y') where it has a singularity of the prescribed type. In order to study the behavior of large $|x|$ it is convenient to make some further transformations on (2.14). We note that,

$$\operatorname*{Res}_{w=w_0}\left\{-\frac{1}{\pi}\,\frac{(K+\gamma)\cosh\gamma y\cosh\gamma y'\,e^{-\gamma a}}{\gamma(K\cosh\gamma a-\gamma\sinh\gamma a)}\right\}=-\frac{1}{\pi}\,\frac{(K+\gamma)\,e^{-\gamma_0 a}}{w_0\sinh\gamma_0 a}\;\frac{\cosh\gamma_0 y\cosh\gamma_0 y'}{1+\frac{2\gamma_0 a}{\sinh 2\gamma_0 a}}\;;$$

thus we obtain, by shifting the contour, C,

$$G(x,y,x',y')=\frac{1}{2\pi}K_0\left(k\sqrt{(x-x')^2+(y-y')^2}\right)+\frac{1}{2\pi}K_0\left(k\sqrt{(x-x')^2+(y+y')^2}\right)$$

$$-\frac{i(K+\gamma)\,e^{-\gamma_0 a}}{w_0\sinh\gamma_0 a}\;\frac{\cosh\gamma_0 y\cosh\gamma_0 y}{1+\frac{2\gamma_0 a}{\sinh 2\gamma_0 a}}\;e^{iw_0|x-x'|}$$

$$-\frac{1}{\pi}\operatorname{Re}\int_{0\,C'}^{\infty}\frac{(K+\gamma)\cosh\gamma y\cosh\gamma y}{\gamma(K\cosh\gamma a-\gamma\sinh\gamma a)}\,e^{iw_0|x-x'|}\,dw$$

(2.15)

where C' is a contour lying in the first or fourth quadrant according as $x-x'$ is >0 or <0. It follows, using (2.10) that,

$$G\sim-\frac{i(K+\gamma_0)\,e^{-\gamma_0 a}}{w_0\sinh\gamma a}\;\frac{\cosh\gamma_0 y\cosh\gamma_0 y'}{1+\frac{2\gamma_0 a}{\sinh 2\gamma_0 a}}\;e^{iw_0|x-x'|}\quad\text{as } |x|\to\infty\,. \tag{2.16}$$

Now the equation (2.2) has, in addition to the real roots $\pm\gamma_0$, an infinity of imaginary roots $\pm i\rho_n$, $n=1,2,\ldots$. Consequently, although we make no use of the fact, the above transformation could be used to obtain an expansion of the integral term in a series of the functions.*

In the case of infinite depth, the transformation may be carried somewhat further in order to yield an expression for G involving only real integrals. For convenience, we change the coordinate system so that y is measured vertically downward from the free surface. Then (2.15) becomes,

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* $\cos\rho_n(y-y')\,e^{\rho_n|x-x'|}$, $\cos\rho_n(y+y')\,e^{-\rho_n|x-x'|}$.

$$G(x,y,x',y') = K_0\left(k\sqrt{(x-x')^2+(y-y')^2}\right) - \frac{2iK}{w_0}\, e^{-K(y+y')}\, e^{iw_0|x-x'|}$$

$$-\frac{1}{\pi}\,\mathrm{Re}\int_{0\atop C'}^{\infty} \frac{(\gamma+K)}{\gamma(K-\gamma)}\, e^{-\gamma(y+y')}\, e^{iw(x-x')}\, dw \quad , \qquad (2.17$$

where now C' consists of the positive imaginary axis except for a small semi-circle in the right half plane about the singularity at $\gamma = 0$, i.e., $w = ik$. Computing the real part one is led to,

$$G(x,y,x',y') = \frac{1}{2\pi}\, K_0\left(k\sqrt{(x-x')^2+(y-y')^2}\right) + \frac{1}{2\pi} K_0\left(k\sqrt{(x-x')^2+(y+y')^2}\right)$$

$$+\frac{2iK}{w_0}\, e^{-K(y+y')}\, e^{iw(x-x')}\,.$$

$$-\frac{2K}{\pi}\int_k^{\infty}\left[\frac{\gamma\sin\gamma(y+y') + K\cos\gamma(y+y')}{\gamma(\gamma^2+K^2)}\, e^{-\tau(x-x')}\right] d\tau \qquad (2.18)$$

with $\gamma = \sqrt{\tau^2-k^2}$ and principal values of the integrals are meant, and where use has been made of (2.13).

3. Wave Generation by Moving Partitions

As a first application we consider the generation of water waves in a basin by means of a vertical partition. It is assumed that the partition is divided into sections along its length, each of which may be moved with an independent amplitude and phase. If the sections are sufficiently small, the distribution of horizontal velocity of the partition may be given effectively an arbitrary shape. We suppose the entire partition to be sufficiently long that we may neglect end effects. The problem is to determine the motion, and the thrust needed to obtain that motion, which should be given the partition in order to produce a prescribed wave motion.

The solution is contained in the following result which is essentially a generalization of the work of Havelock[3] and Kennard[4].

Theorem: The function $\phi(x, y)$ defined by,

$$\phi(x, y) = -2 \int_0^{\infty} f(y') \, G(x, o, x', y') \, dy' , \tag{3.1}$$

is a solution of the problem,

a) $\phi_{xx} + \phi_{yy} - k^2 \phi = 0$ in $0 < x < \infty$, $0 < y < a$

b) $\phi_y = 0$ on $y = 0$, $x > 0$

c) $\phi_y - K\phi = 0$ on $y = a$, $x > 0$

d) $\phi \to A \cosh \gamma_0 y \; e^{i w_0 x}$ as $x \to \infty$

e) $\lim\limits_{x \to 0^+} \dfrac{\partial \phi}{\partial x} = f(y)$ for $0 < y < a$ and $f(y)$ continuous in $0 < y < a$.

Proof: The first three properties follow immediately from those of G ;

(d) follows from (2.16) with,

$$A = + \frac{2i(K+\gamma)}{w_0 \sinh \gamma_0 a} \; \frac{e^{-\gamma_0 a}}{1 + \frac{2\gamma_0 a}{\sinh 2\gamma_0 a}} \int_0^a f(y') \cosh \gamma_0 y' \, dy' . \tag{3.2}$$

From (2.14) it follows that,

$$\lim_{x \to 0^+} \frac{\partial \phi}{\partial x} = \frac{1}{\pi} \lim_{x \to 0} \int_0^a f(y') \frac{\partial}{\partial x} K_0\left(k\sqrt{(x-x')^2 + (y-y')^2}\right) dy' .$$

On making use of (2.11) and the continuity of f we can reduce this further to,

$$\lim_{x \to 0^+} \frac{\partial \phi}{\partial x} = \frac{1}{\pi} \lim_{x \to 0^+} f(y) \int_{-\infty}^{+\infty} \frac{x \, dx}{x^2 + (y-y')^2} .$$

Then we make use of the fact that,

$$\int_{-\infty}^{+\infty} \frac{x \, dx}{x^2 + (y-y')^2} = \begin{cases} \pi & \text{for } x > 0 \\ -\pi & \text{for } x < 0 \end{cases}$$

to obtain $\lim\limits_{x \to 0^+} \dfrac{\partial \phi}{\partial x} = f(y)$. This completes the proof and yields in addition the useful corollary

Corollary: If $\phi(x, y) = -2 \int_0^a f(y') \, K_0\left(k\sqrt{(x-x')^2 + (y-y')^2}\right) dy'$ with $f(y)$ continuous,

$$\lim_{x \to 0^+} \frac{\partial \phi}{\partial x} = f(y) , \qquad \lim_{x \to 0^-} \frac{\partial \phi}{\partial x} = -f(y) . \tag{3.3}$$

Formula (3.3) is a special case of a more general theorem which we will not develop here. Integrals of the above type represent "source layers" analogous to those occurring in ordinary potential theory, and formula (3.3) is merely a statement of the jump condition at such a layer.

We are now in a position to consider the wave generation problem. To simplify the analysis we assume the generator is of the plunger type so that $f(y) = B$, a constant. We denote by $v(y, z, t)$ the horizontal velocity of the partition and by η_∞ the surface elevation at infinity. Noting that,

$$\eta = \frac{1}{g} \frac{\partial \Phi}{\partial t} \Big|_{y=0} \tag{3.4}$$

our theorem enables us to consider several cases.

<u>Case (1)</u> B real,

$$v_1^c = B \cos \sigma t \cos kz, \qquad \eta_\infty^{1,c} = \frac{2\sigma B/g}{w_0 \left(1 + \frac{2\gamma_0 a}{\sinh 2\gamma_0 a}\right)} \cos(w_0 x - \sigma t) \cos kz$$

$$v_1^s = B \cos \sigma t \sin kz, \qquad \eta_\infty^{1,s} = \frac{2\sigma B/g}{w_0 \left(1 + \frac{2\gamma_0 a}{\sinh 2\gamma_0 a}\right)} \cos(w_0 x - \sigma t) \sin kz$$

<u>Case (2)</u> B imaginary,

$$v_2^c = B \sin \sigma t \cos kz, \qquad \eta_\infty^{2,c} = \frac{2\sigma B/g}{w_0 \left(1 + \frac{2\gamma_0 a}{\sinh 2\gamma_0 a}\right)} \sin(w_0 x - \sigma t) \cos kz$$

$$v_2^s = B \sin \sigma t \sin kz, \qquad \eta_\infty^{2,s} = \frac{2\sigma B/g}{w_0 \left(1 + \frac{2\gamma_0 a}{\sinh 2\gamma_0 a}\right)} \sin(w_0 x - \sigma t) \sin kz$$

<u>Case (3)</u> Superposition of v_1^c and v_2^s .

$$v = v_1^c + v_2^s = B \cos(kz - \sigma t) \tag{3.5}$$

$$\eta_\infty = \eta_\infty^{1,c} + \eta_\infty^{2,s} = \frac{2\sigma B/g}{w_0 \left(1 + \frac{2\gamma_0 a}{\sinh 2\gamma_0 a}\right)} \cos(w_0 x - kz - \sigma t) \tag{3.6}$$

The first two cases represent situations in which the sections of the partition move with the same phase but varying amplitudes. The resulting wave motion, sufficiently far from the partition, will form a short-crested wave system propagating outward.

In the third case, on the other hand, the sections move with the same amplitude but differing phases. The resulting system is now a plane wave moving outward at an angle, $\tan^{-1}\frac{k}{w}$, to the x-axis, with frequency σ and wave length $\frac{2\pi}{\gamma_0}$.

If μ_0 denotes the amplitude of the motion of the partition, we obtain from (3.5) and (3.6),

$$\frac{|\eta_\infty|}{\mu_0} = \frac{2K}{w_0}\,\frac{1}{1+\frac{2\gamma_0 a}{\sinh 2\gamma_0 a}} \qquad (3.7)$$

for the amplitude, $|\eta_\infty|$, of the plane wave and in case (3). Now the items we are at liberty to specify are σ and k, hence we take as parameters $Ka = \frac{\sigma^2 a}{2\pi g}$, and ka. For calculations the important parameters are Ka and $\frac{k}{w_0} = \beta$, the latter of which, as we have seen, measures the direction of propagation of the plane wave.

For water of infinite depth, $\gamma_0 = K$, and thus $\frac{Ka}{2\pi}$ is the ratio of the depth to deep water wave length of long-crested waves of frequency σ. This being the case, we can enter Wiegel's[5] tables with $\frac{Ka}{2\pi}$ taking the place of d/L_0.

A few results are indicated in Figure 1. In Figure 2 the corresponding case of a flapper type generator, which is hinged at the bottom, is shown. It is noted that the wave-maker becomes increasingly less efficient as one attempts to increase the propagation angle. In fact, as θ approaches 45° the ratio in (3.7) approaches 0. If one attempts to produce greater angles one merely succeeds in generating so-called "edge waves" which vanish exponentially away from the generator. A similar situation involving a submerged circular

cylinder is considered in a recent paper by Ursell[6].

To obtain the pressure at the partition we can use

$$p(\theta, y, z, t) = \rho g \frac{\partial \Phi}{\partial t}\Big|_{x=0} \quad .$$

We obtain then,

$$p = -\rho g \sigma \operatorname{Re}\left\{ iB \int_0^a G\,(0, y, 0, y')\,dy'\, e^{-i\sigma t} \right\} \begin{matrix}\cos\\ \sin\end{matrix}\; kz\,,$$

an integral which could be obtained by quadrature.

4 Reflection from a Strip

As a second application we consider the reflection of short crested waves from a strip of finite width rigidly fixed in the free surface. Mathematically the problem is the following

$$\phi_{xx} + \phi_{yy} - k^2\phi = 0 \qquad \text{in} \quad 0 < y < a \tag{4.1}$$

$$\phi_y = 0 \qquad \text{on} \quad y = 0 \tag{4.2}$$

$$\phi_y - K\phi = 0 \qquad \text{on} \quad y = a \qquad |x| > b \tag{4.3}$$

$$\phi_y = 0 \qquad \text{on} \quad y = a \qquad |x| < b \tag{4.4}$$

$\phi - A\cosh \gamma_0 y\, e^{iw_0 x}$ satisfies a condition of form (2.9). (4.5)

We remark at this stage that the analysis of Fritz John may be carried over almost intact to yield a fairly general existence theorem for the diffraction problem formulated in section two. This theorem, however, demands that the obstacle surface be perpendicular to the free surface at their intersection, a condition which is not satisfied in the present problem. Our construction will in itself constitute such an existence theorem for the present case.

Consider the function,

$$\psi(x', y') = \int_{-b}^{+b} f(x)\,G(x, a, x', y')\,dx + A\cosh \gamma_0 y'\, e^{iw_0 x} \tag{4.6}$$

where $f(x)$ is a function we will determine later. From (2.16) we find,

$$\psi(x',y') \sim A\cosh\gamma_0 y' e^{iw_0 x'} - i\frac{(K+\gamma_0)e^{-\gamma_0 a}}{w_0\sinh\gamma_0 a}\frac{\cosh\gamma_0 y'}{1+\frac{2\gamma_0 a}{\sinh 2\gamma_0 a}}e^{iw_0 x'}\int_{-b}^{+b} f(x)e^{-iw_0 x}dx$$

$$= A\cosh\gamma_0 y' e^{iw_0 x'} + T_1\cosh\gamma_0 y' e^{iw_0 x'} \quad \text{as} \quad x \to +\infty\ , \quad (4.7)$$

$$\psi(x',y') \sim A\cosh\gamma_0 y' e^{iw_0 x'} - \frac{i(K+\gamma_0)e^{-\gamma_0 a}}{w_0\sinh\gamma_0 a}\frac{\cosh\gamma_0 y'}{1+\frac{2\gamma_0 a}{\sinh 2\gamma_0 a}}e^{-iw_0 x'}\int_{-b}^{+b} f(x)e^{-iw_0 x}dx$$

$$= A\cosh\gamma_0 y' e^{iw_0 x'} + R\cosh\gamma_0 y' e^{-iw_0 x'} \quad \text{as} \quad x \to -\infty\ . \quad (4.8)$$

Thus $\psi(x',y')$ satisfies (4.5) with R representing a reflected wave and $T_1\cosh\gamma_0 y' e^{iw_0 x'}$ combining with the incident wave to give a transmitted wave. That ψ also satisfies (4.1), (4.2), and (4.3) may be verified by direct differentiation. In order to make $\psi(x',y')$ a solution of our problem it remains to show that $f(x)$ may be chosen in such a way that (4.4) is satisfied.

From the defining equation (2.12) for G we find,

$$\frac{\partial G}{\partial y'}(x,a,x',y') = -\frac{1}{\pi}\int_{0}^{\infty} \frac{\sinh\gamma y'}{K\cosh\gamma a - \gamma\sinh\gamma a}\cos w|x-x'|\,dw\ . \quad (4.9)$$

On differentiation of the identity (2.13) with respect to b we find that (4.9) may be written as,

$$\frac{\partial G}{\partial y'}(x,a,x',y') = +\frac{1}{\pi}\frac{\partial}{\partial y'}\left\{K_0\left(k\sqrt{(x-x')^2+(a-y')^2}\right)\right\}$$

$$-\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{\sinh\gamma y}{K\cosh\gamma a-\gamma\sinh\gamma a}+e^{-\gamma(a-y')}\right]\cos w|x-x'|\,dw\ . \quad (4.10)$$

We now state without proof the following result, part of which was essentially proved in section three.

<u>Lemma 1</u>: The function $\Phi(x',y') = \int_{-b}^{+b} f(x)K_0\left(k\sqrt{(x-x')^2+(a-y')^2}\right)dx$, for $f(x)$ continuous in $-b < x < +b$, is a solution of $\Phi_{x'x'}+\Phi_{y'y'}-k^2\Phi = 0$ in $y' < a$,

is continuous on $y' = a$, $-b < x < +b$ when defined as its limit value, from $y' < a$ and satisfies,

$$\lim_{\substack{y' \to a \\ y' < a}} \frac{\partial \Phi(x, y)}{\partial y'} = \pi f(x) \qquad \text{for} \quad -b < x' < b \; .$$

From this lemma and Equation (4.10) we have,

$$\lim_{y' \uparrow a} \frac{\partial \psi(x', y')}{\partial y'} = f(x') + \int_{-b}^{+b} f(x) K(x, x') dx + i w_0 A \cosh \gamma_0 a \, e^{i w_0 x}$$

where

$$K(x, x') = -\frac{1}{\pi} \int_{c'\,0}^{\infty} \left[\frac{\gamma \sinh \gamma a}{K \cosh \gamma a - \gamma \sinh \gamma a} + 1 \right] \cos w |x - x'| \, dw$$

$$= -\frac{K}{\pi} \int_{c'\,0}^{\infty} \frac{\cosh \gamma a \cos w |x - x'| \, dw}{K \cosh \gamma a - \gamma \sinh \gamma a} \; .$$

Setting $\lim_{y \uparrow a} \frac{\partial \psi(x, y)}{\partial y'} = 0$ gives, therefore, an integral equation for the determination of $f(x)$.

Let us investigate the nature of the kernel, $K(x, x')$. We write,

$$K(x, x') = -\frac{1}{\pi} \int_{c'\,0}^{\infty} \frac{\cos w |x - x'| dw}{1 - \frac{\gamma}{K}} \qquad \frac{1}{\pi K} \int_{c''\,0}^{\infty} \frac{\gamma \cos w (x - x') [\tanh \gamma a - 1]}{(1 - \frac{\gamma}{K} \tanh \gamma a)(1 - \frac{\gamma}{K})} dw$$

where C', C'' consist of the positive real axis except for semi-circles in the lower half plane about the singularities. The first term then contains whatever difficulties may arise for $x = x'$. Proceeding as in an earlier calculation, we find,

$$K(x, x') = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-\sqrt{w^2 + k^2} \, |x - x'|}}{\sqrt{w^2 + k^2}} dw + \cdots - \cdots$$

where the dots indicate terms which remain bounded as $x \to x'$. Thus, according to Equation (2.13), $K(x, x')$ behaves like $K_0(k \, | x - x'|)$ near $x' = x$, and has a logarithmic singularity there.

We are thus confronted with the integral equation,

$$- i w_0 A \cosh \chi_0 a \, e^{i w_0 x} = f(x') + \int_{-b}^{+b} f(x) K(x, x') \, dx \ , \tag{4.11}$$

in which the kernel becomes logarithmically infinite as $x \to x'$. The Fredholm theory may be applied and will yield the existence of a continuous solution of (4.11) provided there exist no non-trivial solutions of the associated homogeneous equation. To see that no such solution exists, we suppose there were one and study its properties. Let $f_0(x')$ be the solution and form the function,

$$\psi_0(x', y') = \int_{-b}^{+b} f_0(x) \, G(x, a, x', y') \, dx \ .$$

From the expression (2.14) for G, and Lemma (1) we conclude that $\psi_0(x', y')$ is continuous in $0 < y' \leq a$, $-b < x' < b$. Moreover $f_0(x')$ satisfies (4.11) with the left hand side set equal to 0, hence,

$$\lim_{y' \uparrow a} \frac{\partial \psi_0 (x', y')}{\partial y'} = 0 \ .$$

Now $\dfrac{\partial G(x, a, x', a)}{\partial y'} - K G(x, a, x', a) = 0$ in $-b < x < +b$ except at $x = x'$ where the difference becomes infinite in such a way that

$$\int_{-b}^{+b} f(x) \left[\frac{\partial G(x, a, x', a)}{\partial y'} - K G(x, a, x', a) \right] dx = f(x') \ .$$

Therefore,

$$\lim_{y' \uparrow a} \left\{ \frac{\partial \psi_0(x', y')}{\partial y'} - K \psi (x', y') \right\} = f_0(x') \ . \tag{4.12}$$

(It is important to note that this relation holds also for the functions ψ and $f(x')$ which are to solve our problem, a fact which makes possible a direct determination of $\psi(x', a)$ once $f(x')$ is known, without recourse to another integration.)

Finally we make use of another lemma which we also state without proof:

Lemma 2: Let $\phi(x, y)$ be a solution of $\phi_{xx} + \phi_{yy} - k^2\phi = 0$ in $0 < y < a$ satisfying the following conditions:

a) $\phi_y = 0$ on $y = 0$

b) $\phi_y - K\phi = 0$ on $y = a$, $|x| > b$

c) $\phi_y = 0$ on $y = a$ $|x| < b$

d) $\lim_{L \to \infty} \left\{ \int_0^a |\phi_x(L,y) - iC\phi(L,y)|^2 dy + \int_0^a |\phi_x(-L,y) - iC\phi(-L,y)|^2 dy = 0 \right.$

e) $\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \phi\phi_n \, dl = 0$ where Γ_ϵ consists of the hemispheres, $(x \pm b)^2 + y^2 = \epsilon$, $y < a$.

Then $\phi(x, y) \equiv 0$ in $0 < y < a$.

The lemma is intuitively obvious, since it merely states that if there is no incident wave and the strip is held rigid, no motion can be produced in the fluid. A proof may be derived from an analogous lemma given by Fritz John.

The function $\psi_n(x', y')$ enters the hypothesis of the lemma, in fact will be continuous at $(\pm b, a)$; thus it is identically 0 in $0 < y' < a$, and by the continuity also for $y' = a$. We conclude $f_0(x') \equiv 0$ in $-b < x' < b$.

It is interesting to note that we may pass to the limit as $k \to 0$, while still retaining the same type of behavior for large $|x|$. The fact does not seem obvious since the behavior of solutions of Laplace's equation at large distances is much different than that of solutions of $\phi_{xx} + \phi_{yy} - k^2\phi = 0$ for $k > 0$. For $k = 0$ we obtain for the potential $\phi(x', y')$,

$$\phi(x', y') = -\frac{1}{\pi} \int_{-b}^{+b} f(x) \left[\int_{0,\,C}^{\infty} \frac{\cosh wy'}{\{K \cosh \gamma a - \gamma \sinh \gamma a\}} \cos w|x-x'| \, dw \right] dx \qquad (4.13)$$

where $f(x)$ satisfies,

$$-iw_0 A \cosh w_0 a e^{iw_0 x'} = f(x') - \frac{K}{\pi} \int_{-b}^{+b} f(x) \left[\int_0^{\infty} \frac{\cosh wa}{K \cosh wa - w \sinh wa} \cos w|x-x'| dw \right] dx \qquad (4.14)$$

w_0 being a solution of $K \cosh w_0 a = w_0 \sinh w_0 a$.

If the depth is infinite the kernel may be evaluated explicitly and yields,

$$-i w_0 A e^{i w_0 x'} = f(x') - \frac{K}{\pi} \int_{-b}^{+b} f(x) \Big[\pi i e^{-iK|x-x'|} + \cos K|x-x'| \; Ci\,(K|x-x'|) + \sin K|x-x'| \left(Si\,(K|x-x'|) - \frac{\pi}{2} \right) \Big] dx \qquad (4.15)$$

where Ci, Si are the integral-cosine and integral-sine functions respectively. The appearance of the term $Ci\,(K|x-x'|)$ confirms our earlier remark that the kernel becomes logarithmically infinite as $x \to x'$.

The existence of a solution having been established, one may turn to the question of obtaining numerical results. An equation of the type (4.11) can be solved in a straightforward way by replacing $f(x)$ by a polynomial approximation and thus obtaining a system of n linear equations for the determination of the value of $f(x)$ at n points. The numerical work involves numerical computation of integrals involving $K(x, x')$ and powers of x. The singularity gives no difficulty as it may be subtracted off and the integrals involving it carried out explicitly. Some work of this type is underway and it is hoped may be presented in the future.

A more elegant, although possibly less useful method is the application of the variational methods of Schwinger[7], which were developed to treat aperture diffraction problems. If Equation (4.11) is multiplied by $f(x')$ and integrated over the strip and use is made of (4.8) we find,

$$\frac{1}{r} = \frac{w_0^2 \sinh \gamma_0 a}{(K+\gamma_0) e^{-\gamma_0 a}} \left(1 + \frac{2 \gamma_0 a}{\sinh 2\gamma_0 a} \right) \frac{A}{R} - \frac{\int_{-b}^{+b} f(x')^2 \, dx' + \int_{-b}^{+b} \int_{-b}^{+b} f(x) f(x') K(x,x') \, dx \, dx'}{\left(\int_{-b}^{+b} f(x') e^{i w_0 x'} \, dx' \right)^2} \qquad (4.16)$$

where r is essentially the reflection coefficient. Now the quantity on the right hand side of (4.16) is stationary with respect to first order variations of $f(x)$ about its correct value as determined from the integral equation (4.11).

It might be hoped, therefore, that the reflection coefficient would be relatively insensitive to errors in $f(x)$.

More explicit use of the stationary character of $\frac{1}{r}$ may be made as follows. Suppose we expand $f(x)$ in $(-b, b)$ in a Fourier series*,

$$f(x) = \sum_{-\infty}^{+\infty} a_n \; e^{-inx} \quad .$$

If we substitute this expression in (4.11) we obtain,

$$\left(2b \sum_{-\infty}^{+\infty} a_n \, a_{-n} + \sum\sum_{-\infty}^{+\infty} a_m \, a_n \; C_{mn}\right) r = \left(\sum_{-\infty}^{+\infty} a_n \; B_n\right)^2$$

where

$$C_{mn} = \iint_{-b}^{+b} e^{-inx} \; e^{-imx'} K\,(x, x')\, dx\, dx' \quad , \qquad B_n = \int_{-b}^{+b} e^{-inx} \; e^{iw_0 x} \, dx \quad .$$

Differentiating with respect to a_n and noting that r is stationary,

$$\left(-2\, a_{-m} + \sum_{-\infty}^{+\infty} a_n \; C_{mn}\right) r = 2 B_m \left(\sum_{-\infty}^{+\infty} a_n \;\; B_n\right) \qquad m = 0, \pm 1, \pm 2, \ldots$$

Defining constants, D_p, by

$$a_p = \frac{2}{r} \; D_p \left(\sum_{-\infty}^{+\infty} a_n \; B_n\right) = \frac{2\, C_0}{r} \; D_p \qquad p = 0, \pm 1, \pm 2, \ldots , \qquad (4.17)$$

we obtain finally

$$r = 2 \sum_{-\infty}^{+\infty} D_p \;\; B_p \qquad (4.18)$$

$$D_{-m} + \sum_{-\infty}^{+\infty} C_{mn} \; D_n = B_m \qquad m = 0, \pm 1, \pm 2, \ldots \qquad (4.19)$$

We have thus transformed the problem into solving an infinite system of linear equations, (4.19). Once the D_n's are found, the reflection coefficient can be computed from (4.18). In theory one could also obtain $f(x)$.

* The choice of functions $\{e^{\pm inv}\}$ for the expansion of $f(x)$ is not essential to the method. It suggests itself, however, since for certain values of w_0, the set $\{B_n\}$ reduces to a single term.

Substituting the series for $f(x)$ with a_p replaced by $\frac{2C_0}{r} D_p$ in the integral equation fixes the constant C_0 and thus determines a_p. For the case of diffraction of sound waves through a circular aperture, the scheme has proved quite successful, and it was found that only a few terms of the infinite series suffice to give quite accurate approximations.

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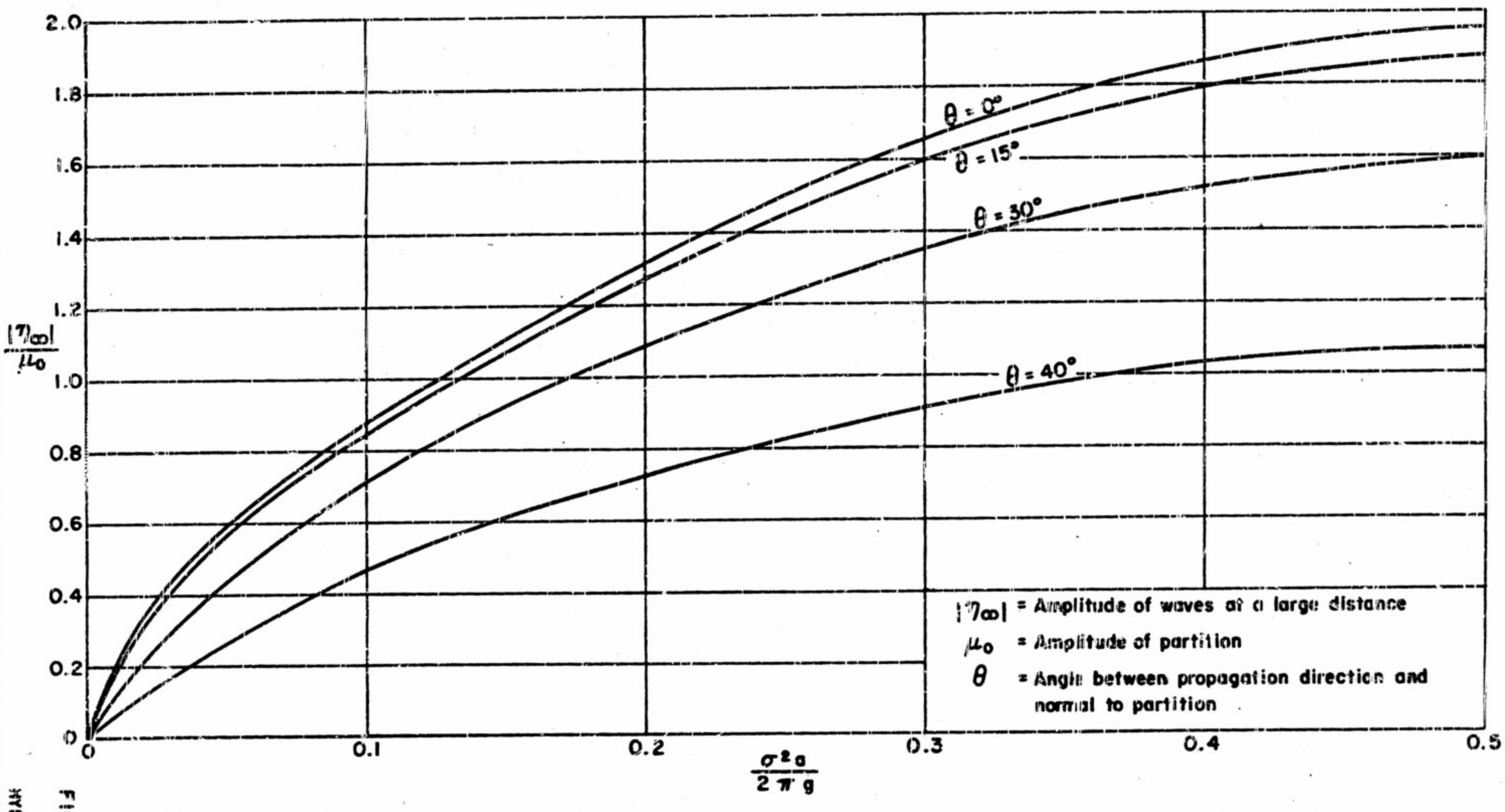

PRODUCTION OF WAVES AT AN ANGLE BY A VERTICAL PARTITION OF PLUNGER TYPE

FIGURE 1

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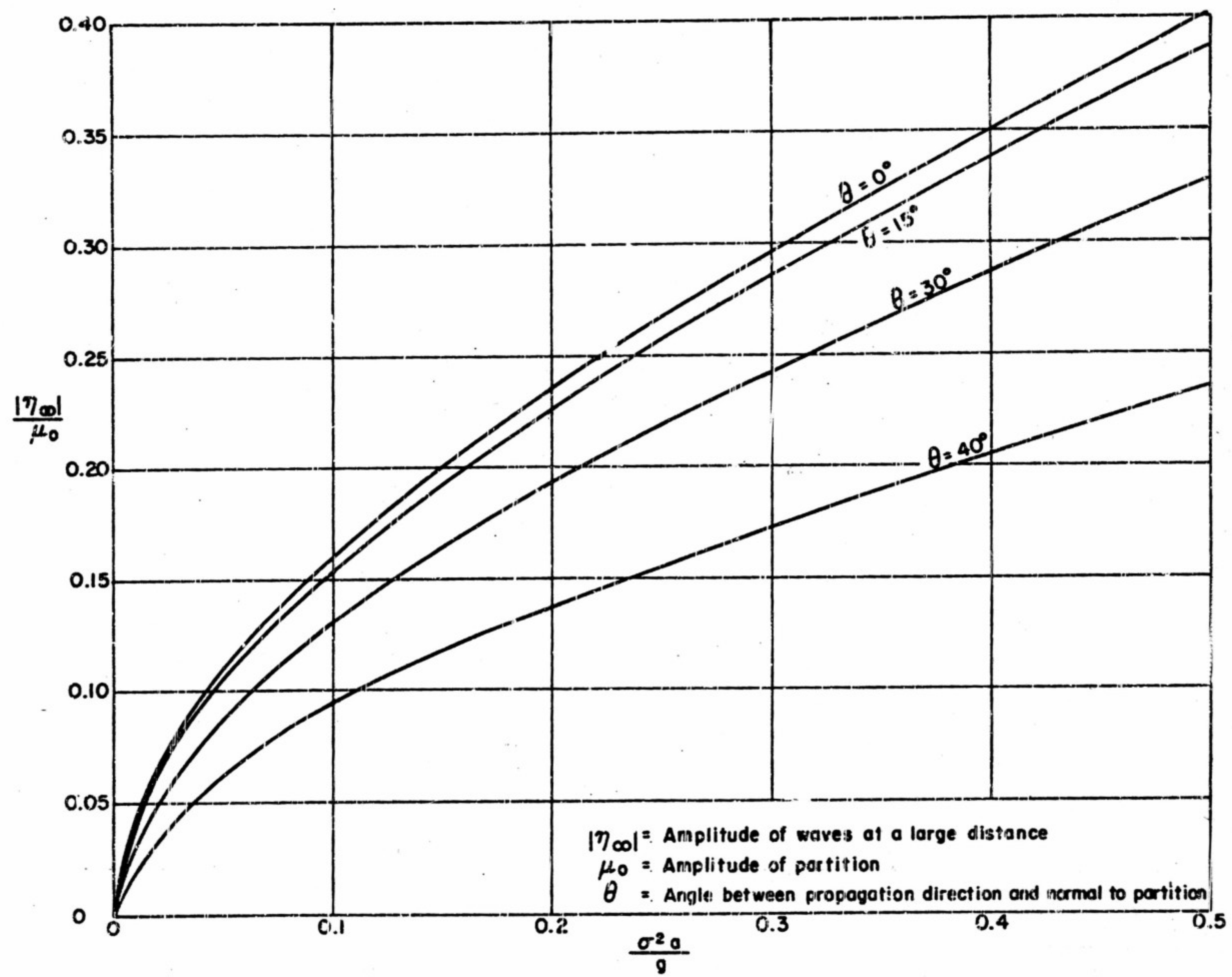

PRODUCTION OF WAVES AT AN ANGLE BY A VERTICAL PARTITION OF FLAPPER TYPE

FIGURE 2

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